F. anal. 13-1. 2025

- info – groups signup
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A. R.-int.s and integrable func'ns

Rem. 1: (SS Appendix)

a) R. int. satifies the usual properties ... $\int f+g = \int f + \int g$ ...

b) f, g R.-int., $c \in \mathbb{R}$, $\varphi \in C(\mathbb{R})$

$\Rightarrow$ $f+g$, cf, $f \cdot g$ and $\varphi(f)$ are all R.-int.

c) f cont. (last time), g piecewise cont. (PC) $\Rightarrow$ f, g R-int.

d) f bnd., non-decreasing, h bnd. variation (BV) $\Rightarrow$ f, h R. int.

[h BV $\Leftrightarrow$ $h = f - g$; f, g bnd., non-decreasing]

e) Jump discontinuities in R.-int. f'ns:

- Finitely many: PC – func'ns
- ∞ – many: non-decreasing f'n: $f(x) = \sum_{r \in \mathbb{Q}}^{\infty} \frac{1}{n^2} \chi_{[0,\infty)}(x-r)$

where $\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$, and indicator f'n $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ (Exercise 1)

f) Not R.-int. on $[-1,1]$:

- $f(x) = x^{-\frac{1}{2}}$ (not bnd) [improper int. $\exists$!!]
- $f(x) = \chi_{\mathbb{Q} \cap [-1,1]}$: $\mathcal{U}(P,f) = 1$, $\mathcal{L}(P,f) = 0$ $\forall P$

Step f'n: $s : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$)

$$s(x) := \sum_{j=1}^{N} a_j \, \chi_{[x_{j-1}, x_j)}(x)$$

Obs. I:

$$\int_a^b s(x)\,dx = \sum_{j=1}^{N} a_j \,(x_j - x_{j-1})$$

$$(*) \qquad \mathcal{U}(P,f) = \int_a^b f^*(x)\,dx, \quad f^* = s \text{ with } a_j = \sup_{x \in [x_{j-1}, x_j]} f(x)$$

$C^k(U)$: $u, Du, \dots, D^k u \in C(U)$, $k \geq 1$

Thm. I: (Approximation by step/C/C^k func'ns)

Let X be step f'ns, $C([a,b])$, or $C^k([a,b])$.

For any R.-int. f on $[a,b]$, $\exists \{f_n\} \subset X$ such that

(i) $|f_n(x)| \leq \|f\|_\infty$, and

(ii) $\int_a^b |f(x) - f_n(x)|\,dx \xrightarrow[n \to \infty]{} 0$

Pf.: 1) X = step f'ns.

Take P_n s.t. $\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) < \frac{1}{n}$,

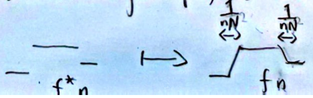
and step f'n f_n^* s.t. $\int_a^b f_n^* = \mathcal{U}(P_n, f)$. (see $(*)$).

Then

$\|f_n^*\|_\infty \leq \|f\|_\infty$,

$$\int_a^b |f_n^* - f| \overset{f_n^* \geq f}{=} \int_a^b (f_n^* - f) = \mathcal{U}(P_n, f) - \underbrace{\int f}_{\geq \mathcal{L}(P_n, f)} < \frac{1}{n}$$

2) $X = C^k$: Smooth out jumps, $f_n^* \mapsto f_n \in C^k$

$k = 0$:  p.w. lin.

$$\|f_n\|_\infty \leq \|f_n^*\|_\infty \leq \|f\|_\infty$$

f_n, f_n^* differ on N-1 intervals of length $\frac{1}{nN}$:

$$\Rightarrow \int_a^b |f_n - f_n^*| \leq 2\|f\|_\infty \cdot N \cdot \frac{1}{nN}$$

Hence

$$\int_a^b |f_n - f| \leq \int |f_n - f_n^*| + \int |f_n^* - f| \xrightarrow[n \to \infty]{} 0$$

$\square$

Rem. 2: $f : [a,b] \to \mathbb{C} \Rightarrow \int_a^b f := \int_a^b \mathrm{Re}\, f + i \int_a^b \mathrm{Im}\, f$

Rem. 3: Periodic functions on $\mathbb{R}$

f L-per : $f(x+L) = f(x) \quad \forall x \in \mathbb{R}$

$f : [0,L] \to \mathbb{R}$ (or $\mathbb{C}$) $\Rightarrow$ F. series $S(f)(x) := \sum_{n=-\infty}^{\infty} a_n e^{\frac{2\pi i n x}{L}}$ is L-per.

$\tilde{f}$ L-per. extension of f : $f = \tilde{f}$ on $[0,L]$, $\tilde{f}$ L-per.

Obs: $S(f) = S(\tilde{f}) \overset{\text{if conv.}}{=} \tilde{f}$

ter

B. L^p-norms and spaces

$f : \mathcal{U} \to \mathbb{R}$ (or $\mathbb{C}$)

$$\|f\|_p = \left(\int_{\mathcal{U}} |f(x)|^p \, dx\right)^{\frac{1}{p}}, \quad p \in [1,\infty)$$

$$\mathcal{R}^p(\mathcal{U}) := \{ f \text{ R.-int} : \|f\|_p < \infty \}$$

Rem. 3: (a) $\mathcal{R}^p$ vec. sp., but $\|\cdot\|_p$ not norm on $\mathcal{R}^p$!

[$\|f\|_p = 0 \Leftrightarrow f = 0$ "except on a set of measure 0]

the $\int f$ does not change if you modify f in some pt's

(b) R^p could not be complete:

$$\int_0^1 |x^{-\frac{1}{2}} - \underbrace{x^{-\frac{1}{2}}\chi_{(\frac{1}{n},1]}(x)}_{\in R^1}| \xrightarrow[n\to\infty]{} 0 \quad \text{but } \underset{\text{not bnd.}}{x^{-\frac{1}{2}}} \notin R^1([0,1])$$

(c) In fact: $R^1 = R^p = R^\infty$, $p \in (1,\infty)$, since $\|f\|_p^p \le \|f\|_\infty^p \cdot |U|$ ($\int_U dx$)

R^p not so usefull. Fix: Introduce Lebesgue measure and int. and Lebesgue sp's $L^p(U)$

$$L^p(U) := \{ f \text{ Lebegue measurable} ; \|f\|_p < \infty \}$$
$$(L^\infty : \|f\|_\infty = \operatorname*{ess\,sup}_{x\in U} |f(x)|)$$

<u>Rem. 4:</u>

a) $\|\cdot\|_p$ (w. L.-int.) is a norm on L^p, <u>when</u> we identify func'ns equal except on a set of (L.-) <u>measure 0</u> (a null set)

So: $f \overset{\text{in } L^p}{=} g$ if $f(x) \ne g(x)$ only for $x \in N$, N a <u>null set</u>

b) N_k null set $\Rightarrow \bigcup_{k=1}^\infty N_k$ null set (SS App.) $\Rightarrow \mathbb{Q}$ null set.

b) $L^p(U)$ is a Banach sp., ($L^p \ne L^q$, $p \ne q$; $\operatorname*{ess\,sup}_{x\in U} |f(x)|$)

~~Lem. 1: $1 \le p \le q \le \infty$, U <u>bnd.</u> ($|U| < \infty$),~~

~~a) $\|f\|_p \le C(1 + \|f\|_q)$, C indep. of f~~ Need $\|f\|_p \le c\|f\|_q$?? No ??

~~b) $L^\infty(U) \subset L^q(U) \subset L^p(U) \subset L^1(U)$~~

Hölder ??

R^p

$\|f - f_n\|_p \le (2\|f\|_\infty)^{p-1} \|f - f_n\|_1 \to 0$

~~Pf.:~~

~~a) $|x|^p \le 1 + |x|^q \; \forall x \Rightarrow \int |f|^p \le \int_U 1 + \int |f|^q$~~

~~b) follows from a) when $q < \infty$~~ □

<u>Prop. 1:</u> Hölder's ineq.:

$f \in L^p(U),\ g \in L^q(U);\ p, q \in [1,\infty],\ \frac{1}{p} + \frac{1}{q} = 1 \quad (\frac{1}{\infty} = 0)$

$\Rightarrow \|f \cdot g\|_1 \le \|f\|_p \cdot \|g\|_q$

<u>Rem. 5:</u>

a) $p = q = 2 \Rightarrow$ Cauchy-Schwarz in(·g)$_1$ =

$|(f,g)_2| = |\int f \cdot \bar{g}| \le \|f \cdot g\|_1 \le \|f\|_2 \cdot \|g\|_2$

b) $p = 1,\ q = \infty$: $\int |f \cdot g| \le \|f\|_\infty \int |g| = \|f\|_\infty \|g\|_1$

c) U bnd, $p \le q \Rightarrow \|f\|_p^p = \|1 \cdot |f|^p\|_1 \le \|1\|^p_{1-\frac{q}{p}} \cdot \||f|^p\|_{\frac{q}{p}}$

<u>Cor. 1:</u> $1 \le p \le q \le \infty$, U bnd. $(|U| < \infty)$

a) $\|f\|_p \le c\|f\|_q \quad , \quad c = \|1\|^{\frac{1}{p}}_{1-\frac{p}{q}}$

b) $L^\infty(U) \subset L^q(U) \subset L^p(U) \subset L^1(U)$

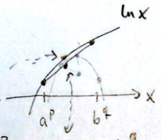
"<u>Pf.:</u>" a) Rem. 2 c).

b) Follows from a) $(q < \infty)$ and $\square$ Rem. 2 b) w. $g = 1$

<u>Pf. of Prop. 1:</u>

dropped

1) Young's ineq: $a \cdot b \le \frac{a^p}{p} + \frac{b^q}{q}$



[$t = \frac{1}{p}$, $1 - t = \frac{1}{q}$

ln concave

$\Rightarrow \ln(t a^p + (1-t) b^q) \ge t \ln a^p + (1-t) \ln b^q$

convex combination

$= \underbrace{tp}_{=1} \ln a + \underbrace{(1-t)q}_{=1} \ln b = \ln(a \cdot b) \Rightarrow e^{\ln(-)} \ge e^{\ln ab} = a \cdot b$]

2) $a = \frac{|f(x)|}{\|f\|_p}$, $b = \frac{|g(x)|}{\|g\|_p} \overset{1)}{\underset{\int - dx}{\Longrightarrow}} \frac{\int |f \cdot g|}{\|f\|_p \cdot \|g\|_q} \le \frac{1}{p} \frac{\int |f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{\int |g|^q}{\|g\|_q^q}$

$= \frac{1}{p} + \frac{1}{q} = 1$ $\square$

Approx. of f R-int. in $\|\cdot\|_p$-norm, $p \in [1,\infty)$:

$\exists \{f_n\}_n$ step / C / C^k - func'ns s.t.

$\|f_n\|_\infty \le \|f\|_\infty$, $\|f_n - f\|_p \to 0$ $\quad (f_n \xrightarrow{L^p} f)$

$$\Big[\ \|f_n - f\|_p^p \le \underbrace{\|f_n - f\|_\infty^{p-1}}_{\le 2\|f\|_\infty} \underbrace{\|f_n - f\|_1}_{\to 0} \xrightarrow{\text{Thm. 1}} 0\ \Big]$$

More generally:

<u>Thm. 2:</u> (Approximation by step f'ns/ C/C^k func'ns)

For any $f \in L^p(U)$, $\exists \{f_n\} \subset C(U)$ (or $C^k(U)$ or step f'ns) such that

$$\|f_n - f\|_p \xrightarrow[n\to\infty]{} 0$$

<u>Rem. 6:</u>

rakk ikke

a) Need some assumptions on U, (ok for intervals)

b) Can modify f_n to be 0 near bnd'ry of U

F. anal. 16.1.2025

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A. F.-series and per.-f'ns

$g : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is L-periodic if

$$g(x+L) = g(x) \quad \forall x \in \mathbb{R}$$

F-series of $f : [a, b] \to \mathbb{R}$ (or $\mathbb{C}$): $L = b-a$,

$$f(x) \sim \underbrace{S(f)(x)}_{\text{F.-series}} := \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\frac{2\pi i n x}{L}}, \quad \hat{f}(n) = \frac{1}{L}\int_a^b f(x) e^{-\frac{2\pi i n x}{L}} dx$$

L-per. extension of f: $\tilde{f} : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) s.t.

(i) $\tilde{f}(x) := f(x)$, $x \in [a, b]$, (ii) $\tilde{f}$ $\underset{b-a}{\overset{\shortparallel}{L}}$-per.

Obs. 1:

(a) $S(f)(x)$ is L-per. and $S(f) \text{ "=" } \tilde{f}$ (if conv.)

Rem. 1:

(a) $f : [a, b] \to \mathbb{R}$ (or $\mathbb{C}$) L-per. $\overset{g(x) = f(a + \frac{L}{2\pi}x)}{\Longrightarrow}$ $g : [0, 2\pi] \to \mathbb{R}\ (\mathbb{C})$ 2π-per.

(b) We identify 2π-per. f'ns f with f'ns F on the circle $\mathbb{S}$:

$$f(\theta) = F(e^{i\theta}), \quad \theta \in \mathbb{R}$$

Notation: $C^k(\mathbb{S}) = C^k_{per}(\mathbb{R}) := \{f : f, Df, \ldots, D^k f \in C(\mathbb{R}), 2\pi\text{-per.}\}$

Obs. 1: a) $f \in C^k(\mathbb{S}) \iff f \in C^k([0, 2\pi])$ "no jump at $x = 0, 2\pi$"

and $D^j f(0^+) = D^j f(2\pi^-)$, $j = 0, 1, \ldots, k$.

B. Uniqueness of F.-series

$$f(x) = g(x),\ x \in [a, b] \ \underset{??}{\overset{ok}{\underset{\Longleftarrow}{\Longrightarrow}}} \ \hat{g}(n) = \hat{f}(n),\ n \in \mathbb{Z}$$

Problem $\Leftarrow$): $(\hat{f} - \hat{g})(n) = \frac{1}{L}\int_{-\frac{a}{2}}^{\frac{b}{2}} (f-g)(x)\, e^{-\frac{2\pi i n x}{L}}\, dx = 0$

$\Rightarrow f = g$ except some pt.'s

But ok where $f-g$ cont.:

Thm. 1: f R.-int. on $[a,b]$, $\hat{f}(n) = 0\ \forall n \in \mathbb{Z}$, f cont. at x_0

$$\Rightarrow f(x_0) = 0$$

Cor. 1: $f \in C([\frac{a}{2}, \frac{b}{2}])$, $\hat{f}(n) = 0\ \forall n \in \mathbb{Z} \Rightarrow f = 0$ in $[a,b]$

Rem. 2: $f \in L^1([a,b])$, $\hat{f}(n) = 0\ \forall n \in \mathbb{Z} \Rightarrow f = 0$ in $L^1(a,b)$,

and hence $f(x) = 0$ except for $x \in N$, N measure 0.

Pf. Thm. 1:

1) Assume $x_0 = 0$, $L = 2\pi$, $f: [-\pi, \pi] \to \mathbb{R}$

[if not: $f(x) \mapsto g(y) = f(\frac{2\pi}{L} y + x_0)$]

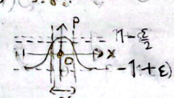
2) Assume by contradiction, $f(0) \neq 0$, say $f(0) > 0$:

Cont. at 0: $\exists\, \delta \in (0, \frac{\pi}{2})$ s.t.

(1) $|x| < \delta \Rightarrow f(x) > \frac{f(0)}{2}\ (>0)$

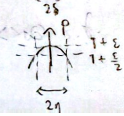
Def. $p(x) := \varepsilon + \cos x$, with $\varepsilon > 0$ s.t.

(2) $\delta \le |x| \le \pi \Rightarrow |p(x)| < 1 - \frac{\varepsilon}{2}$



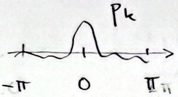
Take $0 < \eta < \delta$ s.t.

(3) $|x| < \eta \Rightarrow p(x) \ge 1 + \frac{\varepsilon}{2}$



Obs:

- f R.-int. $\Rightarrow \|f\|_\infty < \infty$
- $p_k(x) := (p(x))^k$ trig. poly., peak at $x=0$, $p_k(0) \nearrow \infty$ as $k \to \infty$



11:03

Since $\hat{f}(n) = 0 \ \forall n$,

$$\int_0^{2\pi} f(x)\, p_k(x)\, dx \overset{p_k \text{ trig. poly.}}{=} \sum_{n=-N_k}^{N_k} c_n \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$(*) \qquad = \sum_{-N_k}^{+N_k} c_n\, 2\pi \underbrace{\hat{f}(n)}_{=0} \overset{\text{assumption}}{=} 0 \quad \forall k.$$

On the other hand:

$$\left| \int_{\delta \le |x| \le \pi} f(x)\, p_k(x)\, dx \right| \overset{(2)}{\le} 2\pi \|f\|_\infty \left(1 - \frac{\varepsilon}{2}\right)^k \xrightarrow[k \to \infty]{} 0,$$

$$\int_{\eta \le |x| < \delta} f(x)\, p_k(x)\, dx \overset{(1)}{\ge} 0, \qquad \cos x \ge 0,\ |x| < \delta \le \frac{\pi}{2}$$

$$\int_{|x| < \eta} f(x)\, p_k(x)\, dx \underset{(3)}{\overset{(1)}{\ge}} 2\eta \frac{f(0)}{2} \left(1 + \frac{\varepsilon}{2}\right)^k \xrightarrow[k \to \infty]{} \infty.$$

Then

$$\int_{-\pi}^{\pi} f(x)\, p_k(x)\, dx \xrightarrow[k \to \infty]{} \infty, \text{ contradicting } (*).$$

Hence $f(0) = 0$ and Thm. 1 ok.

3) $f : [-\pi, \pi] \to \mathbb{C}$:

$$f = u + iv \Rightarrow u = \frac{1}{2}(f + \bar{f}), \quad v = \frac{1}{2i}(f - \bar{f})$$

$$\hat{f}(n) = 0 \ \forall n \Rightarrow \begin{cases} \hat{u}(n) \overset{\text{chk.}}{=} \frac{1}{2}\left(\hat{f}(n) + \overline{\hat{f}(-n)}\right) = 0 \ \forall n \\ \hat{v}(n) = \frac{1}{2i}\left(\hat{f}(n) - \overline{\hat{f}(-n)}\right) = 0 \end{cases}$$

$$\overset{2)}{\Longrightarrow} u = 0 = v_{\text{at } x_0} \Rightarrow f = 0 \text{ at } x_0 \qquad \square$$

C. Unif. conv. of F.-series

Thm. 2: $f \in C([-a,b])$, $\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$

$$\Rightarrow S_N(f)(x) \xrightarrow[N\to\infty]{\text{unif.}} f(x) \text{ on } [-a,b]$$

Pf.: $([a,b] = [-\pi,\pi])$

1) $|\hat{f}(n) e^{inx}| \leq |\hat{f}(n)| =: M_n$, $\sum_{n=-\infty}^{\infty} M_n < \infty$ (by assumption)

$$\underset{\text{M-test}}{\overset{\text{Weierstrass}}{\Longrightarrow}} \sum_{-N}^{N} \hat{f}(n) e^{inx} \xrightarrow[N\to\infty]{} \sum_{-\infty}^{\infty} \hat{f}(n) e^{inx} =: g(x) \text{ abs. and unif.}$$

2) $g \in C([-\pi,\pi])$ [unif. lim. of cont. f'ns], and

$\hat{g}(n) = \hat{f}(n)$ $\forall n$:

$$\left[2\pi \hat{g}(n) = \int_{-\pi}^{\pi} \Big(\sum_{m=-\infty}^{\infty} \hat{f}(m) e^{imx}\Big) e^{-inx} \overset{\substack{\Sigma \text{ unif. conv.} \\ \downarrow}}{=} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \hat{f}(m) e^{i(m-n)x} \right.$$

$$= \lim_{N\to\infty} \sum_{m=-N}^{N} \hat{f}(m) \cdot \underbrace{\int_{-\pi}^{\pi} e^{i(m-n)x}\, dx}_{= \begin{cases} 2\pi, & m=n \\ 0, & m\neq n \end{cases}} = 2\pi \hat{f}(n)$$

3) By Thm. 1, $g = f$ $\forall x \in [-\pi,\pi]$ $\square$

When does $\sum |\hat{f}(n)|$ conv. ?

Regularity of f $\Rightarrow$ decay of $\hat{f}(n)$ as $n \to \infty$:

Lem. 1: $f \in C(S) \cap C^1([-\pi,\pi])$

(a) $\widehat{f'}(n) = in \hat{f}(n)$

(b) $|\hat{f}(n)| \leq \frac{1}{n} \|f'\|_{L^1} \left(\leq \frac{1}{n} 2\pi \|f'\|_\infty\right)$

Pf. $2\pi \hat{f}(n) = \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = \left[f(x) \frac{1}{-in} e^{-inx} \right]_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} f'(x) e^{-inx} = 2\pi \hat{f}(n)$ $\square$

Pf.:

(a) $2\pi \hat{f}(n) = \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx \overset{\text{i.b.p.}}{=} \underbrace{\cancel{\left[f(x) \frac{1}{-in} e^{-inx}\right]_{-\pi}^{\pi}}}_{\in C(\$)} \to 0 + \frac{1}{in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx$

$= 2\pi \frac{1}{in} \widehat{f'}(n)$

(b) $|\hat{f}(n)| \overset{(a)}{=} |\frac{1}{n}| \, |\widehat{f'}(n)| \le \frac{1}{n}\frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} |f'(x)| \cdot 1 \, dx}_{= \|f'\|_1} \le \frac{1}{n} \|f'\|_\infty \cdot 2\pi$ □

Cor. 2: $f \in C^{k-1}(\$) \cap C^k([-\pi,\pi])$

(a) $\widehat{f^{(k)}}(n) = (in)^k \hat{f}(n)$

(b) $|\hat{f}(n)| \le \frac{1}{n^k} \|f^{(k)}\|_1 \le \frac{1}{n^k} \|f^{(k)}\|_\infty \cdot 2\pi$

Pf: Exercise 2

Cor. 3: $f \in C^1(\$) \cap C^2([-\pi,\pi])$

$\Longrightarrow S_N(f)(x) \xrightarrow{\text{unif.}} f(x)$ on $[-\pi,\pi]$

Pf.: Cor. 2 $\Rightarrow |\hat{f}(n)| \le \frac{K}{n^2}$ ~~$\Rightarrow \Sigma$~~

Hence $\Sigma |\hat{f}(n)| < \infty$ and Thm. 2 $\Rightarrow$ Cor. 3 □

12:02

Rem. 1: Unif. conv. also holds if $f \in C^1(\$)$, or Lipschitz, or even

~~(Later:~~ $\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty$ for $\alpha \in (\frac{1}{2}, 1]$ (f Hölder cont.)

$\Longrightarrow S_N(f) \xrightarrow{\text{unif}} f$

~~Hence~~ (Later)